A 400 liter tank initially holds 40 liters of brine containing 1 gram of salt per liter. **SCORE: _____ / 12 PTS**
Brine containing 2 grams of salt per liter starts flowing into the tank at 6 liters per minute.
At the same time, the well-mixed solution leaves the tank at 4 liters per minute.

[a] Find the amount of salt in the tank t minutes after the brine starts flowing into the tank, assuming the tank does not overflow.
HINT: Simplify all fractions as soon as possible.

$x(t)$ = AMOUNT OF SALT IN TANK @ TIME t

$\frac{dx}{dt}$ = RATE SALT IN − RATE SALT OUT

= RATE BRINE IN * CONCENTRATION OF SALT IN BRINE

− RATE SOLUTION OUT * CONCENTRATION SALT IN SOLUTION

↳ = $\dfrac{\text{AMOUNT OF SALT IN TANK}}{\text{VOLUME OF SOLUTION IN TANK}}$

$$\frac{dx}{dt} = \underbrace{6}_{\frac{1}{2}} \cdot \underbrace{2}_{\frac{1}{2}} - \underbrace{4}_{\frac{1}{2}} \cdot \frac{\overbrace{x}^{1}}{\underbrace{40+(6-4)t}_{1}} = 12 - \frac{2x}{20+t}, \quad \underbrace{x(0) = 40 \cdot 1 = 40}_{1}$$

$$1 \quad \underbrace{\frac{dx}{dt} + \frac{2}{20+t}x = 12}_{} \rightarrow \mu = e^{\int \frac{2}{20+t}dt} = e^{2\ln|20+t|} = \underbrace{(20+t)^2}_{}, \ 1$$

$$1 \quad \underbrace{(20+t)^2 \frac{dx}{dt} + 2(20+t)x = 12(20+t)^2}_{} \qquad \text{CHECK: } 1 \ \underbrace{\frac{d}{dt}(20+t)^2 = 2(20+t)}_{} \checkmark$$

$$1 \quad \underbrace{(20+t)^2 x = \int 12(20+t)^2 dt + C}_{}$$

$$\underbrace{= 4(20+t)^3 + C}_{}$$

$$1 \quad \underbrace{x = 4(20+t) + C(20+t)^{-2}}_{}$$

$$40 = 4(20) + \frac{C}{20^2}$$

$$C = -16000$$

$$\underbrace{x = 4(20+t) - 16000(20+t)^{-2}}_{}, \ 1\tfrac{1}{2}$$

[b] **BONUS: (the case in which the rate at which brine enters, and solution leaves, are not constant)** **SCORE: _____ / 3 PTS**

If brine enters the tank at $I(t)$ liters per minute, and solution leaves the tank at $O(t)$ liters per minute, and all other information remains the same as above, write **BUT DO NOT SOLVE** an initial value problem for the amount of salt in the tank t minutes after the brine starts flowing into the tank, assuming the tank does not overflow.

$$\frac{dx}{dt} = \boxed{2I(t) - O(t)\frac{x}{40 + \underbrace{\int_0^t (I(u) - O(u))\,du}_{}}} \quad 1 \quad 2$$

The population model $\frac{dP}{dt} = kP^{1+c}$ (where $k, c > 0$) is called a doomsday equation, because solutions of the equation go to infinity in a finite time. (In contrast, the exponential model in class goes to infinity in infinite time.) **SCORE: _____ / 9 PTS**

[a] Solve the initial value problem $\frac{dP}{dt} = kP^{1+c}, \quad P(0) = P_0$. You must give an **explicit** solution, which will involve k and c.

$$\int P^{-1-c}\,dP = \int k\,dt \quad 1$$

$$-\tfrac{1}{c}P^{-c} = kt + b \quad 1$$

$$P^{-c} = -ckt + b$$

$$P = (-ckt+b)^{-\frac{1}{c}} \quad 2$$

$$P_0 = b^{-\frac{1}{c}} \longrightarrow b = P_0^{-c}$$

$$P(t) = (P_0^{-c} - ckt)^{-\frac{1}{c}} \quad 2 \quad = \frac{1}{(P_0^{-c} - ckt)^{\frac{1}{c}}}$$

[b] Find the time at which doomsday occurs (the population goes to infinity).

$$2 \quad (P_0^{-c} - ckt)^{\frac{1}{c}} = 0$$

$$P_0^{-c} - ckt = 0$$

$$1 \quad t = \frac{1}{ckP_0^c}$$

OR $t = \frac{P_0^{-c} - P^{-c}}{ck}$ 1

OR

$$\lim_{P\to\infty} \frac{P_0^{-c} - P^{-c}}{ck} = \frac{1}{ckP_0^c} \quad 2$$

GRADE USING ONLY 1 VERSION ☆

The population model $\frac{dP}{dt} = (a - b\ln P)P$ is known as the Gompertz differential equation (where $a, b > 0$). **SCORE: _____ / 9 PTS**

Solve that differential equation with the initial condition $P(0) = P_0$. You must give an **explicit** solution, which will involve a and b.

$$\int \frac{1}{P(a-b\ln P)}\,dP = \int dt \quad 2$$

$$u = a - b\ln P$$

$$-\tfrac{1}{b}\ln|a - b\ln P| = t + c \quad 3$$

$$a - b\ln P = Ce^{-bt}$$

$$a + Ce^{-bt} = b\ln P$$

$$\ln P = \frac{a}{b} + Ce^{-bt}$$

$$P = e^{\frac{a}{b} + Ce^{-bt}} \quad 2$$

$$P_0 = e^{\frac{a}{b} + C}$$

$$C = \ln P_0 - \frac{a}{b}$$

$$P = e^{\frac{a}{b} + (\ln P_0 - \frac{a}{b})e^{-bt}} \quad 2$$